

Are networks with more edges easier to synchronize? *

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Abstract. In this paper, the relationship between the network synchronizability and the edge distribution of its associated graph is investigated. First, it is shown that adding one edge to a cycle definitely decreases the network synchronizability. Then, since sometimes the synchronizability can be enhanced by changing the network structure, the question of whether the networks with more edges are easier to synchronize is addressed. It is shown by examples that the answer is negative. This reveals that generally there are redundant edges in a network, which not only make no contributions to synchronization but actually may reduce the synchronizability. Moreover, an example shows that the node betweenness centrality is not always a good indicator for the network synchronizability. Finally, some more examples are presented to illustrate how the network synchronizability varies following the addition of edges, where all the examples show that the network synchronizability globally increases but locally fluctuates as the number of added edges increases.

Keywords. Complex network, Complementary graph, Synchronizability, Edge addition.

1 Introduction and problem formulation

Systems composing of dynamical units are ubiquitous in nature, ranging from physical to technological, and to biological fields. These systems can be naturally described by networks with nodes representing the dynamical units and links representing the interactions

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among them. The topologies of such networks have been extensively studied and some common architectures have been discovered [1, 2]. The small-world property, for example, characterized by short average distance and high clustering among nodes, is one of the most common properties shared by many real networks [3]. More significantly, many networks show high heterogeneity of node connectivity, which typically possesses a power-law distribution, named scale-free networks [4]. It is known that these topological characteristics have strong influence on the dynamics of the structured systems, such as epidemic spreading, traffic congestion, collective synchronization, and so on [5, 6]. From this viewpoint, systematically understanding the network structural effects on their dynamical processes is of both theoretical and practical importance.

Synchronization behavior, in particular, as a widely observed phenomenon in networked systems has received a great deal of attention in the past decades [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Oscillator network models have been commonly used to characterize synchronization behaviors. In this setting, a synchronizability theorem provided by Pecora and Carroll [19] indicates that the collective synchronous behavior of a network is completely determined by the network structure, independent of individual oscillator dynamics, provided that the network coupling strength satisfies some strong conditions. In this framework, it has been found that, compared to regular lattices, small-world networks have remarkably better synchronizability [20]. In contrast to small-world networks, scale-free networks tend to inhibit synchronization, although they have much shorter average distances than small-world networks [21] which are generally deemed to be advantageous for synchronization. Therefore the node betweenness centrality was provided as a good indicator to the network synchronizability [22]. Since the synchronizability is correlated with many topological properties, a natural question is which property is the most significant to the synchronizability? Donetti et al. [23] gave a good answer to this question by an optimization argument. They pointed out that a network with optimized synchronizability should have an extremely homogeneous structure, i.e., the distributions of some fundamental topological properties should be very narrow. Their work provides a constructive approach to the issue of networked synchronization, making a big progress in this area. However, some issues still remain unclear, e.g., what is the most important topological property for the synchronizability? And what is the effect of the connectivity density on the synchronizability? In particular, as admitted by the authors, this approach cannot theoretically guarantee to find the optimal solution.

Motivated by the above works, this paper focuses on the relationships between the network synchronizability and the edge distribution of the associated graph. The effects of the connection patterns of graphs on the synchronizability are analyzed both theoretically and numerically. It is found that adding an edge to a cycle of size $N \geq 5$ definitely decreases the network synchronizability, but the synchronizability may be improved by changing the

cyclic structure. However, a further example shows that, by arbitrarily optimizing the network structures, networks with more edges are not necessarily easier to synchronize. This implies that there are redundant edges in the network with respect to synchronization.

Consider a dynamical network consisting of N coupled identical nodes, with each node being an n -dimensional dynamical system, described by

$$\dot{x}_i = f(x_i) - c \sum_{j=1}^N a_{ij} H(x_j), \quad i = 1, 2, \dots, N, \quad (1)$$

where $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbb{R}^n$ is the state vector of node i , $f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth vector-valued function, constant $c > 0$ represents the coupling strength, $H(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called the inner linking function, and $A = (a_{ij})_{N \times N}$ is called the outer coupling matrix or topological matrix, which represents the coupling configuration of the entire network. This paper only considers the case that the network is diffusively connected, i.e., A is irreducible and its entries satisfy

$$a_{ii} = - \sum_{j=1, j \neq i}^N a_{ij}, \quad i = 1, 2, \dots, N.$$

Further, suppose that, if there is an edge between node i and node j , then $a_{ij} = a_{ji} = -1$, i.e., A is a Laplacian matrix. In this setting, 0 is an eigenvalue of A with multiplicity 1, and all the other eigenvalues of A are strictly positive, which are denoted by

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_N. \quad (2)$$

The dynamical network (1) is said to achieve (asymptotical) synchronization if $x_1(t) \rightarrow x_2(t) \rightarrow \dots \rightarrow x_N(t) \rightarrow s(t)$, as $t \rightarrow \infty$, where, because of the diffusive coupling configuration, the synchronous state $s(t) \in \mathbb{R}^n$ is a solution of an individual node, i.e., $\dot{s}(t) = f(s(t))$.

It is well known that the eigenratio $r(A) = \frac{\lambda_2}{\lambda_N}$ of the network structural matrix A characterizes the synchronizability. The larger the $r(A)$ is, the better the synchronizability will be. The enhancement of the network synchronizability and the relationships between $r(A)$ and the network structural characteristics such as average distance, node betweenness, degree distribution, clustering coefficient, etc., have been well studied [24, 25, 26, 27, 28, 29]. In particular, the graph-theoretical method was used to discuss the network synchronizability in [24, 25]. This paper further investigates the relationship between the network edges and its synchronizability by graph-theoretical tools.

Throughout this paper, for any given undirected graph G , eigenvalues of G mean eigenvalues of its corresponding Laplacian matrix. Notations for graphs and their corresponding Laplacian matrices are not differentiated, and networks and their corresponding graphs are not distinguished, unless otherwise indicated.

2 Adding one edge to a cycle

It has been shown by examples that adding edges can either increase or decrease the network synchronizability [25], and it was found [30] that in scale-free networks where the nodes are coupled symmetrically, if some overloaded edges are removed, the network will become more synchronizable.

In this section, consider adding one edge to a given cycle with N ($N \geq 4$) nodes. In this case, the added edge definitely decreases the synchronizability. To show this, the following lemmas are needed.

Lemma 1 [31, 32] For any given connected graph G of size N , its nonzero eigenvalues indexed as listed in (2) grow monotonically with the number of added edges; that is, for any added edge e , $\lambda_i(G + e) \geq \lambda_i(G)$, $i = 1, \dots, N$.

Lemma 1 shows the eigenvalue changes of graphs due to the addition of edges, but it does not show any information about the eigenratio $r(A)$. Therefore, this eigenratio needs to be studied in more detail.

Lemma 2 [27, 31] For any given connected graph G of size N , its largest eigenvalue λ_N satisfies $\lambda_N \geq d_{\max} + 1$, with equality if and only if $d_{\max} = N - 1$. Further, if G is not a complete graph, then the smallest nonzero eigenvalue of G satisfies $\lambda_2 \leq d_{\min}$. Here d_{\max} and d_{\min} denote the maximum and minimum degrees of G .

Lemma 3 [27] For any cycle C_N with N (≥ 4) nodes, its eigenvalues are given by μ_1, \dots, μ_N (not necessarily ordered as in (2)) with $\mu_1 = 0$ and

$$\mu_{k+1} = 3 - \frac{\sin(\frac{3k\pi}{N})}{\sin(\frac{k\pi}{N})}, \quad k = 1, \dots, N - 1.$$

Lemma 4 Given a connected graph G , if the multiplicity of its smallest nonzero eigenvalue λ_2 is larger than or equal to 2, then adding one edge to G can not change this eigenvalue, i.e., $\lambda_2(G + e) = \lambda_2(G)$.

Proof This lemma follows from the fact that $\text{rank}(\lambda_2 I - (G + e)) \leq \text{rank}(\lambda_2 I - G) + 1$.
□

By the above lemmas, one can get the following result for cycles.

Theorem 1 For any cycle C_N with $N \geq 4$ nodes, adding one edge can not enhance its synchronizability $r(C_N)$; specifically, one has $r(C_4 + e) = r(C_4)$ and $r(C_N + e) < r(C_N)$ ($N \geq 5$).

Proof $r(C_4 + e) = r(C_4)$ holds obviously. For the case of $N \geq 5$, by Lemma 2, one has $\lambda_N(C_N + e) > 4$. But by Lemma 3, $\lambda_N(C_N) \leq 4$. And Lemma 3 shows that the multiplicity of the smallest nonzero eigenvalue λ_2 of C_N is 2. By Lemma 4, $\lambda_2(C_N + e) = \lambda_2(C_N)$. Therefore, $r(C_N + e) < r(C_N)$ for all $N \geq 5$. □

Theorem 1 shows that adding one edge a cycle with $N \geq 5$ nodes definitely decreases the network synchronizability, as shown by the two examples in Figs. 1-5. By simple computation, one obtains that $r(C_5) = \frac{1.3820}{3.6180} = 0.3820$ and $r(C_5 + e\{1, 3\}) = \frac{1.3820}{4.6180} = 0.2993 < r(C_5)$; $r(C_6) = \frac{1}{4} = 0.25$, $r(C_6 + e\{1, 3\}) = \frac{1}{4.4142} = 0.2265 < r(C_6)$ and $r(C_6 + e\{1, 4\}) = \frac{1}{5} = 0.2 < r(C_6)$.

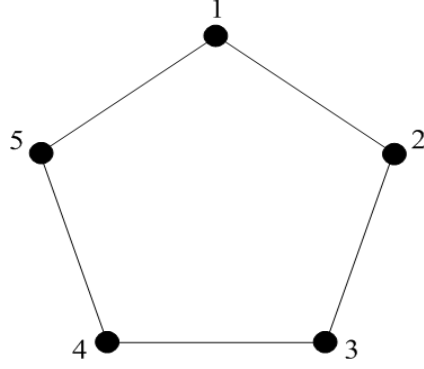


Fig. 1 Graph C_5

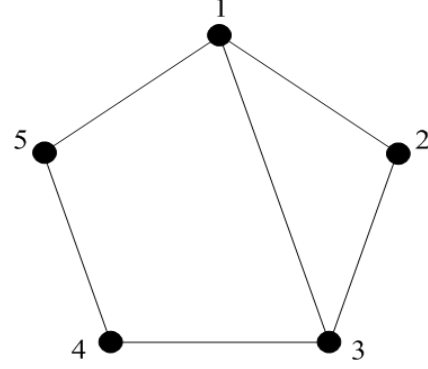


Fig. 2 Graph $C_5 + e\{1, 3\}$

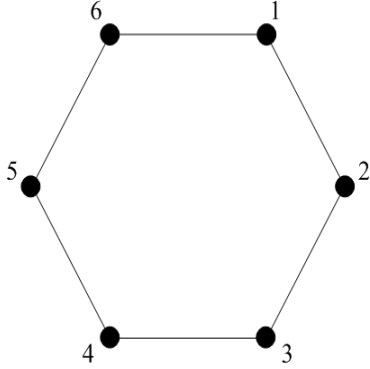


Fig. 3 Graph C_6

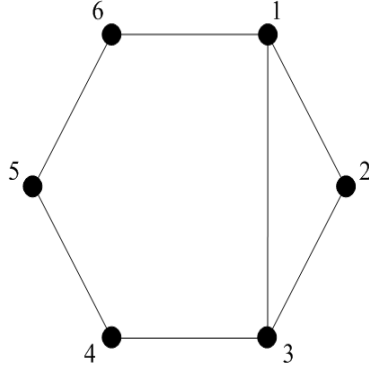


Fig. 4 Graph $C_6 + e\{1, 3\}$

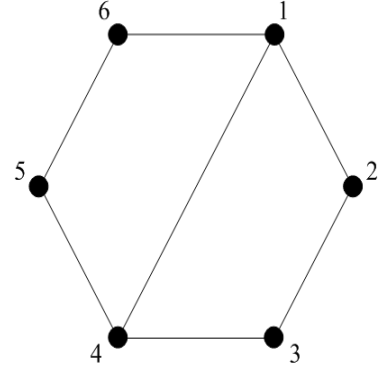


Fig. 5 Graph $C_6 + e\{1, 4\}$

From the above two examples, one can find that the synchronizability of cycles strictly decreases if only one edge is added, and the results vary depending upon where the edge is placed, e.g., $r(C_6 + e\{1, 3\}) > r(C_6 + e\{1, 4\})$. Considering the optimization of network structures, $r(C_6 + e\{1, 3\})$ is still not the best one among all graphs with 7 edges connecting 6 nodes in a cycle, as demonstrated in the next section.

3 Changing the network structure to enhance its synchronizability

It is shown in the above section that adding one edge to a cycle decreases its synchronizability. A further question is whether the synchronizability can be enhanced by changing the network

structure after edge addition. The answer is ‘yes’ in some cases. For example, one can change $C_5 + e\{1, 3\}$ to C_{5o} as in Fig. 6, and $C_6 + e\{1, 3\}$ to C_{6o} as in Fig. 7. Then, $r(C_{5o}) = \frac{2}{5} = 0.4$ and $r(C_{6o}) = \frac{1.2679}{4.7231} = 0.2684$.

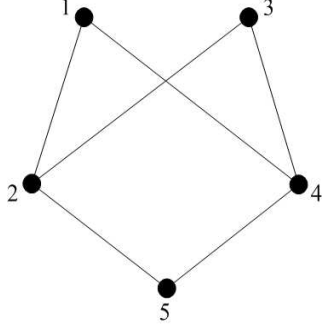


Fig. 6 Graph C_{5o}

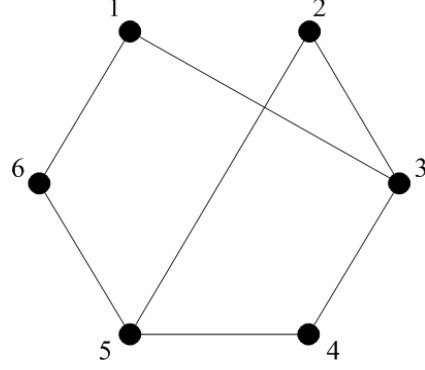


Fig. 7 Graph C_{6o}

Comparing with the graphs in Figs. 1-5, one can see that both the synchronizabilities of C_{5o} and C_{6o} have been improved. In fact, two cycles share a common edge in Figs. 2, 4 and 5. In this case, generally the betweenness centrality is large, or the node-to-node distances are not homogeneous. In comparison, the network structural characteristics are more homogeneous in Figs. 6 and 7. This is consistent with the result of [23]. For simple graphs with a few nodes and edges, as those shown above, one can compute their eigenvalues to find a good structure for the synchronizability. However, for a general graph, how to optimize the network structure toward the best possible synchronizability? Some optimal rules are provided based on an optimizing algorithm in [23]: to have homogeneous degree, node distance, betweenness, and loop distributions. But these rules are observed from simulations, theoretical proofs are not available by now. And, sometimes, these rules are contrary to each other. For example, comparing $C_6 + e\{1, 3\}$ with $C_6 + e\{1, 4\}$, one can find that the cycle of $C_6 + e\{1, 4\}$ is more homogeneous, but the average node distance of $C_6 + e\{1, 3\}$ is smaller. It seems that the importance of these rules should be ordered. Although some rules are provided in [23], optimizing the network structure for better synchronizability is still a hard problem, since it is possible that the optimizing algorithm converges to a suboptimal solution.

Other than the rules for optimization, complementary graphs can be used to characterize the network synchronizability [25]. For a given graph G , the complementary graph of G , denoted by G^c , is the graph containing all the nodes of G and all the edges that are not in G . For eigenvalues of graphs and complementary graphs, the following lemma is useful (see [31, 32] and references therein).

Lemma 5 For any given graph G , the following statements hold:

- (i) $\lambda_N(G)$, the largest eigenvalue of G , satisfies $\lambda_N(G) \leq N$.
- (ii) $\lambda_N(G) = N$ if and only if G^c is disconnected.
- (iii) If G^c is disconnected and has (exactly) q connected components, then the multiplicity of $\lambda_N(G) = N$ is $q - 1$.
- (iv) $\lambda_i(G^c) + \lambda_{N-i+2}(G) = N$, $2 \leq i \leq N$.

For example, the complementary graph of C_{5o} is disconnected (see Fig. 8) and the largest eigenvalue of C_{5o} is 5, the number of nodes. Its smallest nonzero eigenvalue $\lambda_2 = 2$ can be easily obtained by computing the largest eigenvalue of its complementary graph. Further, according to the complementary graph, adding one more edge to graph C_{5o} can not enhance its synchronizability. However, if adding two more edges to C_{5o} , e.g., $e\{1, 5\}$ and $e\{3, 5\}$, then the synchronizability increases to $r = \frac{3}{5}$. The corresponding complementary graph becomes the complementary graph of cycle C_4 (see Fig. 9). Cycle C_4 and its complementary graph are very important in graph theory [32] (see the section below for their further applications).

For a given graph G , if its complementary graph is disconnected and includes two separated graphs G_1 and G_2 , then by Lemma 5 the synchronizability of G is $r(G) = \frac{N - \max\{\lambda_{\max}(G_1), \lambda_{\max}(G_2)\}}{N}$, where N is the number of nodes of G and λ_{\max} denotes the maximum eigenvalue of the corresponding Laplacian matrix. It is well known that the complementary graphs of bipartite graphs are disconnected [25, 33], so the synchronizability of bipartite graphs can be simply analyzed by the above method. Actually, C_{5o} in Fig. 6 is a bipartite graph. Obviously, better understanding and careful manipulation of complementary graphs are useful for enhancing the network synchronizability (see the section below for further applications of complementary graphs).



Fig. 8 Graph C_{5o}^c

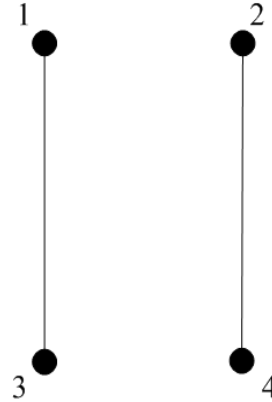


Fig. 9 Graph C_4^c

4 Are networks with more edges easier to synchronize?

For a given graph G , let \mathcal{V} and \mathcal{E} denote the sets of nodes and edges of G , respectively. A graph G_1 is called an induced subgraph of G , if the node set \mathcal{V}_1 of G_1 is a subset of \mathcal{V} and the edges of G_1 are all edges in \mathcal{E} . In this section, subgraphs and complementary graphs are used to discuss network synchronizability.

In the concern of optimizing network structures, an interesting question is whether networks with more edges are easier to synchronize. In order to answer this question, the following lemma is needed.

Lemma 6 For any given graph G , suppose G_1 is its induced subgraph including all nodes of G with the maximum degree d_{max} . If G_1 includes a cycle C_{2k} with even nodes $2k(k \geq 2)$ as an induced subgraph, then the largest eigenvalue of G satisfies $\lambda_N(G) \geq d_{max} + 2$.

Proof By Lemma 3, for any cycle C_{2k} with even nodes, its largest eigenvalue is 4. And since the degree of every its node is 2, -2 must be an eigenvalue of the adjacency matrix of C_{2k} . Let L_1 be the sub-matrix of the Laplacian matrix of G associated with all the nodes in G_1 . By the assumption, one has

$$(d_{max} + 2)I - L_1 = 2I + A(G_1) \not\geq 0,$$

where $A(G_1)$ is the adjacency matrix of G_1 . This implies that the largest eigenvalue of G_1 is larger than or equal to $d_{max} + 2$. Thus, Lemma 1 leads to the result directly. \square

Remark 1 Besides Lemma 2, there are few results on lower bounds of the largest eigenvalue of Laplacian matrices in graph theory [31, 33]. Since networks with good synchronizability always have homogeneous degree distributions, Lemma 6 is very useful for the study of network synchronization.

Theorem 2 For any graph G with 16 edges on 10 nodes, its eigenratio is bounded by $r(G) < \frac{2}{5}$.

Proof If the largest node degree of G is $d_{max} \geq 6$, then the smallest node degree must satisfy $d_{min} \leq 2$. The conclusion follows directly from Lemma 2. In order to have good synchronizability, the degree distribution of G should be homogeneous. Then, first suppose that G has 8 nodes with degree 3 and two nodes with degree 4. In this case, by Lemma 2, the largest eigenvalue of G is $\lambda_{10}(G) > 5$.

In what follows, consider the largest eigenvalue of the complementary graph G^c . By the above discussion, G^c must have 8 nodes with degree 6 and two nodes with degree 5. Suppose G_1 is the subgraph of G^c composing of 8 degree-6 nodes. By direct computing, G_1 must have 19 or 20 edges, and the degree of every its node is at least 4. Hence, G_1^c has 9 or 8 edges and the degree of every its node is at most 3. If the largest eigenvalue of G_1 is 8, i.e.,

G_1^c is disconnected (Lemma 5), then the largest eigenvalue of G^c is larger than or equal to 8. Therefore, the smallest nonzero eigenvalue of G is $\lambda_2(G) \leq 10 - 8 = 2$. By the above discussion, the theorem obviously holds. Hence, suppose both G_1 and G_1^c are connected. Then, G_1 must have a cycle C_4 as an induced subgraph. This holds if and only if G_1^c has C_4^c (see Fig. 9) as an induced subgraph. With only 9 or 8 edges having node degree at most 3, drawing G_1^c directly one can easily reach the conclusion. By Lemma 6, the largest eigenvalue of G^c must be larger than or equal to 8. Repeating the above discussion concludes the proof.

When G has 9 nodes with degree 3 and one node with degree 5, the proof can be similarly completed. \square

Remark 2 Theorem 2 shows that there is not a graph G with 16 edges on 10 nodes whose synchronizability is $r(G) \geq \frac{2}{5}$. However, there does exist a graph Γ_1 with 15 edges on 10 nodes whose synchronizability is $r(\Gamma_1) = \frac{2}{5}$ (see Fig. 10), consistent with the result of [23]. This clearly shows that networks with more edges are not necessarily easier to synchronize. In fact, by the optimal result of [23], $r = \frac{2}{5}$ is the optimal synchronizability for graphs with 15 edges on 10 nodes. For any graph G with 16 edges on 10 nodes, if both G and G^c have cycles with even nodes, then by Lemma 6 and Theorem 2, $r(G) \leq \frac{2}{6} = \frac{1}{3}$. Therefore, adding one more edge definitely decreases the synchronizability. The existence of cycles with even nodes can be easily tested by drawing graphs, so Lemma 6 is very useful for analyzing the synchronizability of homogeneous networks. Actually, the graph shown in Fig. 10 is quite homogeneous in structure [23]. With one more edge being added, such a structure is destroyed. It is therefore easy to understand why adding more edges do not necessarily result in better synchronizability.

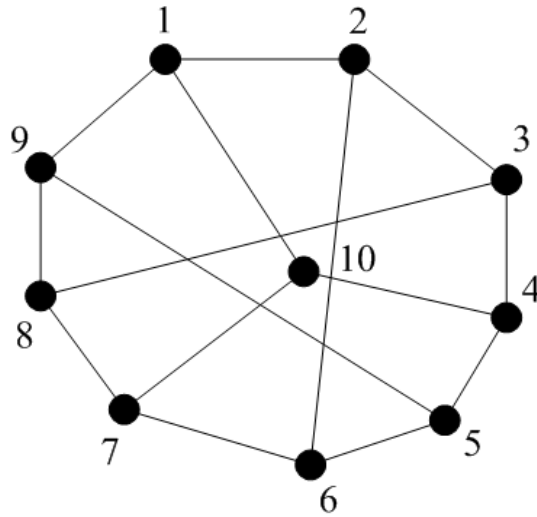


Fig. 10 Graph Γ_1 , $r(\Gamma_1) = \frac{2}{5}$

Remark 3 Fig. 11 shows a new graph Γ_2 with 20 edges on 10 nodes. It also has quite homogeneous structural characteristics as discussed in [23]. In fact, the betweenness centrality

of each node of Γ_1 is 6, larger than that of Γ_2 , 5. But, the synchronizability of graph Γ_2 is worse than that of graph Γ_1 , contrary to the result of [22]. So far, the existing theories [21, 22, 23, 28] can not explain why the synchronizability of Γ_1 is better than that of Γ_2 . This shows the complexity of the relationships between the synchronizability and network structural characteristics. Although Γ_2 has the property of homogeneity, another question is whether there exists another graph with 20 edges on 10 nodes having better synchronizability than that of Γ_1 or Γ_2 ? If the answer is negative, it implies that generally there are many redundant edges in a network with respect to its synchronizability. This kind of questions are still open today.

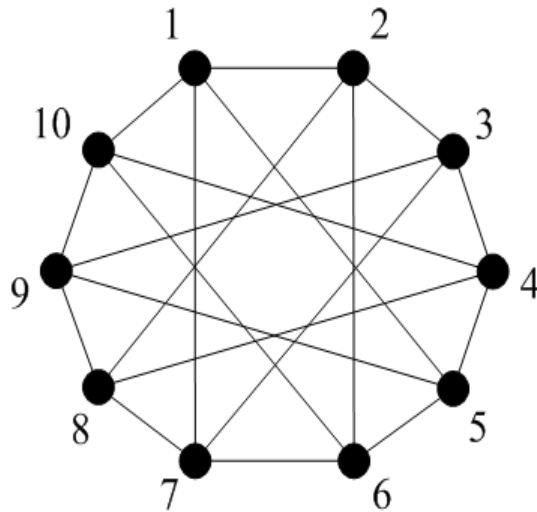
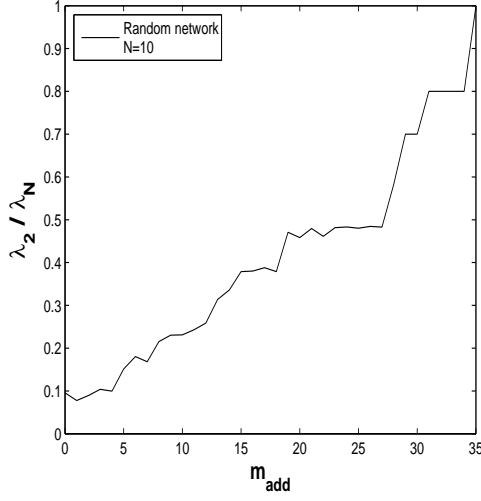


Fig. 11 Graph Γ_2 , $r(\Gamma_2) = \frac{2.7639}{7.2361} \approx 0.382 < r(\Gamma_1) = \frac{2}{5}$

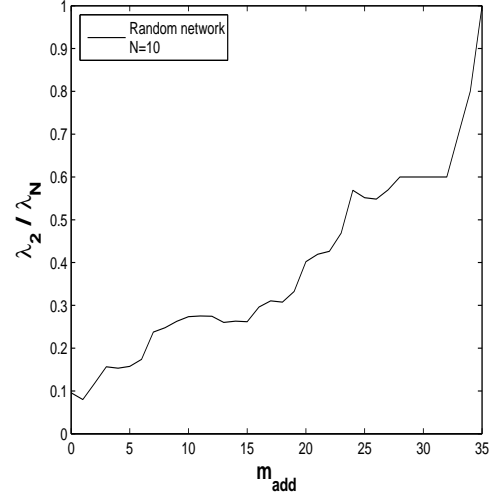
5 Some examples

In this section, some examples are given to show the changes of the synchronizability versus the addition of edges.

Example 1 The synchronizability changes by adding edges to graphs with cycles are shown in Figs. 12 and 13, where their initial graphs are C_{10} and C_{50} , respectively, and m_{add} denotes the number of added edges. The figures in (a)s show the synchronizability changes during the process of adding edges with degree homogeneity (i.e., guaranteeing the node degrees be as homogeneous as possible during edge-adding). The figures in (b)s show the cases corresponding to random edge-adding. Naturally, the corresponding synchronizabilities in (a)s are better than those in (b)s, since degree homogeneity is an important property for networks to achieve good synchronizabilities. In all graphs, it is shown that the synchronizability globally increases but locally fluctuates. According to Theorem 2 and Remark 2, this is the expected phenomenon.

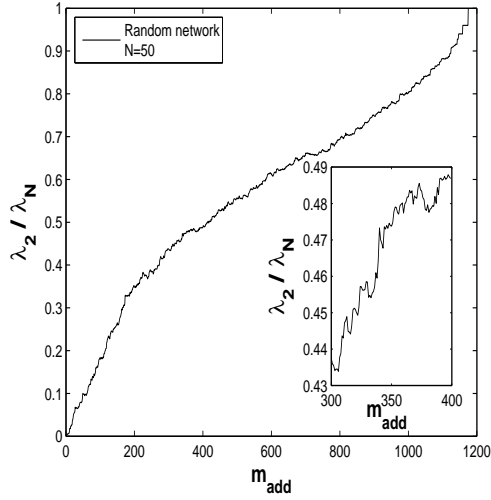


(a) Adding edges with degree homogeneity

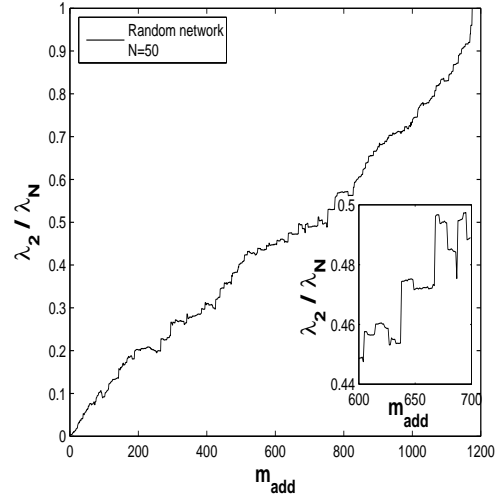


(b) Randomly adding edges

Fig. 12 The synchronizability changes of graphs obtained from C_{10} by adding edges



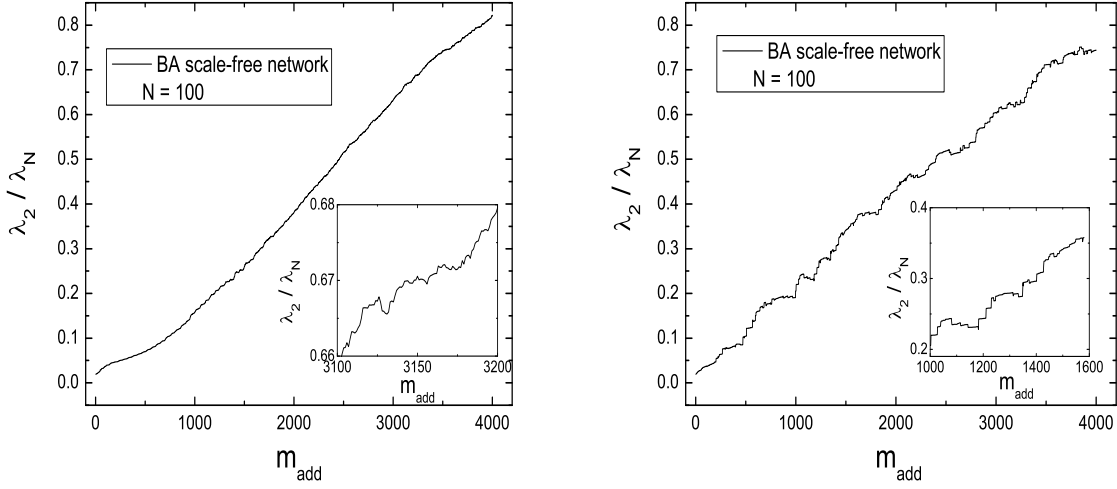
(a) Adding edges with degree homogeneity



(b) Randomly adding edges

Fig. 13 The synchronizability changes of graphs obtained from C_{50} by adding edges

Example 2 The synchronizability changes of graphs obtained from a scale-free graph by randomly adding edges are shown in Fig. 14, for which, the same conclusion can be drawn as in Example 1.



(a) Adding edges with degree homogeneity (b) Randomly adding edges

Fig. 14 The synchronizability changes of graphs obtained from a scale-free graph by adding edges

6 Conclusion

In this paper, the relationship between the network synchronizability and the edge distribution of the associated graphs has been studied. It has been proved that the synchronizability definitely decreases if one edge is being added to a cycle with N ($N \geq 5$) nodes. However, it has also been shown that the synchronizability can be improved by changing the network structure. Further, some examples have shown that some networks with more edges, unexpectedly, have worse synchronizabilities even if the network structures are in some sense optimized. This implies that, for network synchronization, generally there are redundant edges, which do not make any contribution to synchronization but may actually destroy the synchronizability. In addition, an example of a graph with 20 edges on 10 nodes has been provided to show that the existing theories can not explain why it has worse synchronizability than that of a graph with 15 edges on 10 nodes. Some other examples have also been given to show that the network synchronizability globally increases but locally fluctuates due to edge-adding. According to these results, in practical synchronization problems, the synchronizability and the number of communication edges should have a coordinative relation. And one may utilize the redundant edges to improve robustness or other network properties. These kinds of important questions remain open for further research in the future.

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